

Mathematical Aspects of Numerical Solution of Hyperbolic Systems

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Abstract. A number of physical phenomena are described by nonlinear hyperbolic equations. Presence of discontinuous solutions motivates the necessity of development of reliable numerical methods based on the fundamental mathematical properties of hyperbolic systems. Construction of such methods for systems more complicated than the Euler gas dynamic equations requires the investigation of existence and uniqueness of the self-similar solutions to be used in the development of discontinuity-capturing high-resolution numerical methods. This frequently necessitates the study of the behavior of discontinuities under vanishing viscosity and dispersion. We discuss these problems in the application to the magnetohydrodynamic equations, nonlinear waves in elastic media, and electromagnetic wave propagation in magnetics.

1. Introduction

In this paper we discuss the mathematical aspects of the problems originating in the solution of nonlinear systems of hyperbolic partial differential equations. These equations describe a large variety of physical phenomena, such as, gasdynamics, magnetohydrodynamics (MHD), shallow water equations, elasticity equations, etc. Being nonlinear, these systems usually require numerical methods for their solution. Presence of discontinuous solutions motivates the necessity of the development of reliable numerical methods based on the fundamental mathematical properties of hyperbolic systems. Although such methods are rather well developed for the Euler gasdynamic equations in the conservation law form, their extension to more complicated hyperbolic systems is not straightforward. It requires a mathematical justification of the solution uniqueness, a formulation of the selection principles for relevant solutions, and, finally, an investigation of their physical validity. Most of high-resolution methods for gasdynamic equations use the exact or some of the approximate self-similar Riemann problem solutions to determine fluxes through the computational cell surfaces. Similar methods are expected to be developed for various types of hyperbolic systems. In this case we must construct the elementary self-similar solution using only admissible discontinuities (entropy consistent, evolutionary, etc.). Basically the choice of the solution must be made

on the basis of the structure of the solution of the extended problem [14]. All mentioned above makes very important the study of discontinuous solutions behavior under vanishing viscosity and dispersion to create a proper background for the development of high-resolution numerical methods for hyperbolic systems more complicated than the Euler equations of gasdynamics. We discuss several analytical and numerical solutions in the mentioned fields which illustrate the complexity of the selection problem and outline the methods of its solution.

2. High-resolution methods for MHD equations

TVD upwind and symmetric differencing schemes have recently become very efficient tool for solving complex multi-shocked gasdynamic flows. This is due to their robustness for strong shock wave calculations. The extension of these schemes to the equations of the ideal magnetohydrodynamics is not simple. First, the exact solution [16] of the MHD Riemann problem is too multivariant to be used in regular calculations. Second, several different approximate solvers [1], [6], [7], [9], [12], [18], and [23] applied to MHD equations are now at the stage of investigation and comparison. This investigation requires i) determination of a proper slope limiting method in the parameter interpolation procedure necessary to obtain nonoscillatory schemes of the order of accuracy higher than one; ii) development of an efficient entropy correction method necessary to exclude rarefaction shocks; and, finally, iii) solution of the problem of excluding the origin of nonevolutionary solutions in ideal MHD calculations.

The system of governing equations for a MHD flow of an ideal, infinitely conducting, perfect plasma in the Cartesian coordinate system x, y, z with the use of the conventional notations reads (one fluid approximation):

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} + \mathbf{H}_{\text{div}} = \mathbf{0}, \quad (1)$$

where \mathbf{U} is the vector of conservative variables and \mathbf{E} , \mathbf{F} , and \mathbf{G} are the flux vectors.

We introduced here the source term \mathbf{H}_{div} in the form

$$\mathbf{H}_{\text{div}} = \text{div } \mathbf{B} \times \left(0, \frac{B_x}{4\pi}, \frac{B_y}{4\pi}, \frac{B_z}{4\pi}, \frac{\mathbf{v} \cdot \mathbf{B}}{4\pi}, u, v, w \right)^T.$$

This form of the system can be used to satisfy the divergence-free condition by convecting away the magnetic charge from the computational region [23]. Otherwise, any other well-known method can be used to eliminate the magnetic charge.

To determine a numerical flux $\bar{\mathbf{E}} = n_1 \mathbf{E} + n_2 \mathbf{F} + n_3 \mathbf{G}$ normal to the computational cell boundary ($\mathbf{n} = (n_1, n_2, n_3)$ is a unit outward vector normal to the cell surface) one can use the formulas based on the solution of the linearized problem

$$\bar{\mathbf{E}}(\mathbf{U}^R, \mathbf{U}^L) = \frac{1}{2} [\mathbf{E}(\mathbf{U}^L) + \mathbf{E}(\mathbf{U}^R) - S|\Lambda|S^{-1}(\mathbf{U}^R - \mathbf{U}^L)]. \quad (2)$$

Here $S(\bar{\mathbf{U}})$ and $S^{-1}(\bar{\mathbf{U}})$ are the matrices formed by the right and by the left eigenvectors, respectively, of the frozen Jacobian matrix

$$\bar{J} = \frac{\partial \bar{\mathbf{E}}(\mathbf{U}^L, \mathbf{U}^R)}{\partial \mathbf{U}}.$$

The matrix $|\Lambda|$ is a diagonal matrix consisting of the frozen Jacobian matrix eigenvalue moduli. The superscripts R and L denote the values at the right- and at the left-hand side of the cell boundary.

In pure gas dynamics the uniform average vector $\bar{\mathbf{U}}(\mathbf{U}^L, \mathbf{U}^R)$ can be constructed in such a way that the conservation relations on shocks are exactly satisfied. The important peculiarity of the latter method is that, although it gives the solution of the linearized problem, the exact satisfaction of the Rankine–Hugoniot relations on shocks provides their more adequate and sharp resolution [24].

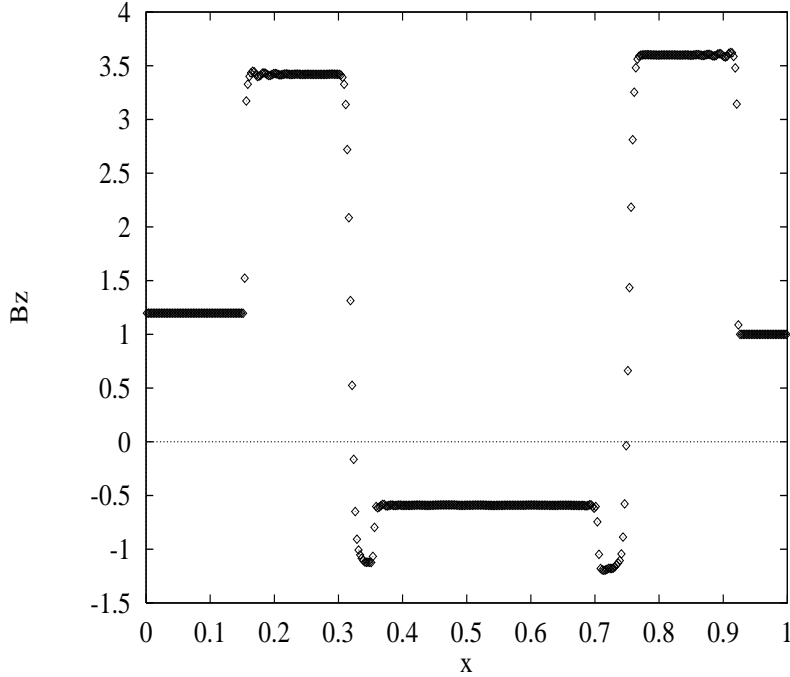
In [7] the MHD numerical Jacobian matrix was used frozen at the point obtained by the arithmetic average between \mathbf{U}^R and \mathbf{U}^L . Thus, it could not guarantee the exact satisfaction of the conservation condition at the jump. This approach belongs to the general conservative Courant–Isaacson–Rees family.

The reason of the former averaging for the MHD equations is explained by the fact that there is no single averaging procedure to find a frozen Jacobian matrix of the system. Another linearization approach is used in [1], [12], [18] in which the linearized Jacobian matrix is not a function of a single averaged set of variables, but depends in a complicated way on the variables on the right- and on the left-hand side of the computational cell surface. In [19] and [20] this procedure was shown to be nonunique. A multiparametric family of linearized MHD approximate Riemann problem solutions was presented that assured an exact satisfaction of the conservation relations on discontinuities. A proper choice of parameters is necessary to avoid physically inconsistent solutions.

It is widely known that the interpolation method used to determine parameter values at the cell surfaces can greatly improve the quality of numerical results. Let us introduce the mesh function $\mathbf{U}_i^n = \mathbf{U}(n\Delta t, i\Delta x)$. We shall use as an example the following interpolation approach [6]:

$$\begin{aligned} \mathbf{U}_{i+1/2}^R &= \mathbf{U}_{i+1}^n - \frac{1}{4}[(1-\eta)\tilde{\Delta}_{i+3/2} + (1+\eta)\tilde{\Delta}_{i+1/2}], \\ \mathbf{U}_{i+1/2}^L &= \mathbf{U}_i^n + \frac{1}{4}[(1-\eta)\tilde{\Delta}_{i-1/2} + (1+\eta)\tilde{\Delta}_{i+1/2}], \\ \tilde{\Delta}_{i+1/2} &= \min \text{mod}(\Delta_{i+1/2}, \omega\Delta_{i-1/2}), \quad \tilde{\Delta}_{i+1/2} = \min \text{mod}(\Delta_{i+1/2}, \omega\Delta_{i+3/2}), \\ \min \text{mod}(x, y) &= \text{sgn}(x) \times \max\{0, \min[|x|, y \text{sgn}(x)]\}. \end{aligned}$$

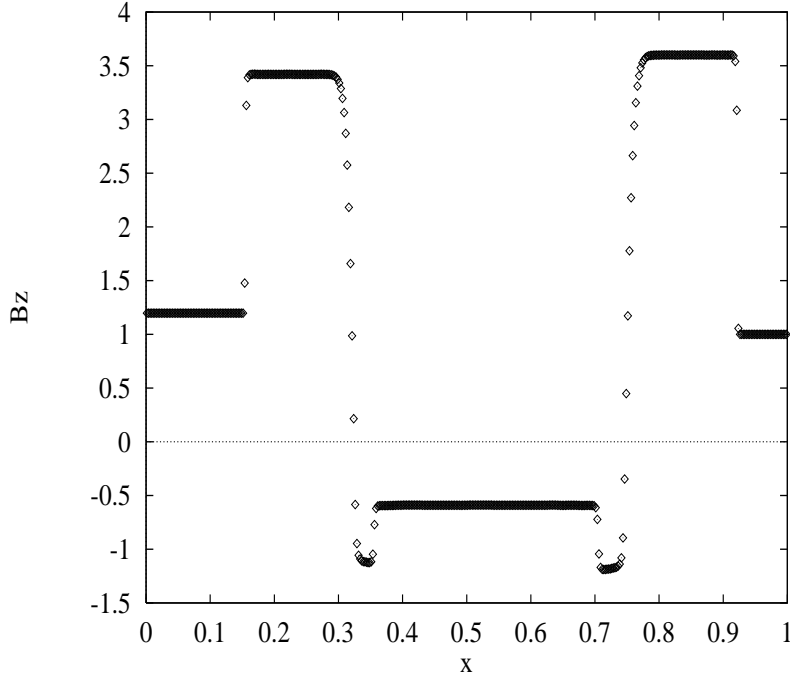
The choice $\eta = -1$ and $\omega = 1$ gives a popular “min mod” method which usually eliminates spurious oscillations near discontinuities. On the other hand, the application of more compressive slope limiters is widely accepted in gas dynamics for finer resolution of contact discontinuities. In MHD the same is important for rotational (Alfvénic) discontinuities. We can suppose the choice $\eta = \frac{1}{3}$ (the third-order upwind-biased interpolation) and $\omega = 2$ to give better results. Consider as

FIGURE 1. B_z distribution for $\eta = 1/3$

an example the MHD Riemann problem with the following initial conditions $(\rho, p, u, v, w, B_y/\sqrt{4\pi}, B_z/\sqrt{4\pi}) = (0.18405, 0.3541, 3.8964, 0.5361, 2.4866, 2.394, 1.197)$ for $x < 0.5$ and $(0.1, 0.1, -5.5, 0, 0, 2, 1)$ for $x > 0.5$ with $B_x \equiv 4$ and the specific heat ratio $\gamma = 1.4$. The solution of this problem contains all types of MHD shocks propagating through the both halves of the computational region separated by the contact discontinuity [6]. The distribution of the B_z component of the magnetic field vector is presented for $t = 0.15$ (400 cells were taken between 0 and 1). One can clearly see in Fig. 1 the numerical noise attendant in this case in the vicinity of strong shocks, which is similar to that being suppressed by the artificial viscosity in [9].

If one decides, however, to choose $\eta = 1$ (three-point central differencing scheme) these oscillations disappear (Fig. 2). This result seems to have no analogue in purely gas dynamic calculations and speaks in favor of the application of the central schemes rather than upwind ones.

Talking about the Roe-type solvers for MHD, it is worth noting that application of the one-dimensional solver for multidimensional calculations even with the use of Powell's technique [23] cannot guarantee the exact satisfaction of the Rankine–Hugoniot conditions due to the presence of the artificial source term.

FIGURE 2. B_z distribution for $\eta = 1$

This makes questionable the necessity of utilization of the very complicated and time consuming algorithms like [19]. Our experience based on the solution of several one-dimensional and two-dimensional test problems shows that the scheme [7] gives essentially the same accuracy.

Another important thing of discussion is that certain initial and boundary value problems can be solved nonuniquely using different shocks or combination of shocks, whereas physically one would expect only unique solutions. The situation differs from that in pure gasdynamics where all entropy increasing solutions are evolutionary and physically relevant. Contrary to pure gasdynamics, in the MHD case the condition of the entropy increase is necessary but not sufficient. Only slow and fast MHD shocks are evolutionary, while intermediate shocks are to be excluded. Although in the ideal MHD a nonevolutionary shock decomposition into evolutionary ones occurs instantaneously under action of an infinitesimal perturbation, this decomposition in the presence of numerical dissipation can require some time which depends on the numerical scheme and grid resolution [6]. From this viewpoint the question arises whether the schemes are applicable which use one-dimensional coplanar Riemann problem solution to determine numerical fluxes at cell interfaces. Among the schemes mentioned above only [9] effectively

incorporates Alfvénic shocks into the flux determination procedure. On the other hand, it is well known that boundary conditions for rotational perturbations in MHD split from the full set of boundary conditions on the shock. That is why, the evolutionarity properties for this kind of perturbations must be checked separately. Does this mean that axisymmetric problems must be solved as three-dimensional or some automatic algorithm can be constructed allowing one to introduce rotational discontinuities in the framework of the two-dimensional statement of the problem? The answer to this question is still open. One should also admit that the destruction time of the latter kind of nonevolutionary waves can be rather long. Although it is clear that shocks unstable to tangential perturbations can be stable in dissipative MHD, their behavior in the case of uncontrollable numerical dissipation is hardly predictable.

In the context of this section we would like to indicate a very simple numerical algorithm which was proposed by N. Pogorelov in [6]. In this approach instead of Eq. (2) we use the formula

$$\begin{aligned}\bar{\mathbf{E}}_{i+1/2,n} &= \frac{1}{2} \left[\bar{\mathbf{E}} \left(\mathbf{U}_{i+1/2}^R \right) + \bar{\mathbf{E}} \left(\mathbf{U}_{i+1/2}^L \right) + \Phi_{i+1/2} \right], \\ \Phi_{i+1/2} &= -\hat{\mathbf{R}}_{i+1/2} \left(\mathbf{U}_{i+1/2}^R - \mathbf{U}_{i+1/2}^L \right),\end{aligned}\tag{3}$$

where $\hat{\mathbf{R}}_{i+1/2}$ is the diagonal matrix with the same elements on its diagonal equal to the spectral radius r (the maximum of eigenvalue magnitudes) of the Jacobian matrix $\frac{\partial \bar{\mathbf{E}}}{\partial \mathbf{U}}$.

This scheme can be treated as a second-order nonoscillatory extension of the Lax–Friedrichs method. It is extremely robust and automatically satisfies the entropy condition, thus allowing one to avoid the application of artificial entropy fix procedures. Several complicated axisymmetric and three-dimensional physical problems (see [21] and [22]) were successfully solved using this scheme.

3. The solution selection problem

We shall describe here several physical problems for which the Riemann problem has nonunique solutions. This nonuniqueness is merely caused by the formulation of the problem, since in the classical physics the future can be predicted uniquely by initial conditions. The oversimplification is connected with the following two circumstances.

Alongside with the hyperbolic system of equations presupposing the presence of discontinuities in its solutions, there exist as a rule a more complete system which takes into account such processes as viscosity, heat conduction, diffusion, finite electric conductivity, etc. This system transforms into the hyperbolic one for large-scale phenomena if certain small terms are neglected. The complete system usually has only continuous solutions which can be supposed unique. The mentioned system, however, has no self-similar solutions depending only on x/t .

Self-similar solutions of this kind can originate as the asymptotics of solutions of appropriate problems for the complete system for $t \rightarrow \infty$. Let us consider as well-selected the self-similar solution which is the asymptotics of some definite problem for the complete system. The suggested selection principle clearly depends on the physical processes which are not taken into account in the simplified hyperbolic system. In the theoretical study it also depends on the selection of the complete system itself.

Note that not only solutions corresponding to discontinuous initial conditions but also solutions with arbitrarily (and not necessarily monotonically) smeared discontinuities in initial conditions have the same asymptotics x/t . The particular specification of this smearing can affect the establishment of one or another asymptotics. This is the second circumstance which influence the correct choice of the self-similar solution.

The first step in the application of the complete system of equations is usually represented by selection among all discontinuities satisfying the conservation laws those “admissible” discontinuities which have a corresponding solution of the discontinuity structure problem described by the complete system. As a discontinuity structure, we imply the solution which represents a continuous variation of values corresponding to the jump, this solution being usually considered one-dimensional and stationary. Note that the latter limitations are not always satisfied for realistic discontinuities. The requirement of existence of this structure provides the entropy nondecrease condition and, if the conservation laws are not sufficient for the discontinuity to be evolutionary, is used to obtain new additional boundary conditions whose satisfaction can make the discontinuity evolutionary. The latter statement was proved for the case of the stationary one-dimensional structure [13] and also for the case of periodicity with respect to time and space variables along the front (cell structure) [14].

In a number of cases the requirement of admissibility of all discontinuities turns out to be sufficient for the solution uniqueness. As an example we can mention the problem of motion of the gas with a complicated equation of state [2] where viscosity and heat conductivity are taken into account in the discontinuity structure. On the other hand, there exist problems for which the requirement of admissibility is insufficient for the uniqueness of the self-similar solution. One of the well-known examples is the decay of an arbitrary jump in a combustible gas mixture. There exist mixtures for which both detonation and slow combustion are possible depending on the ignition method, that is, on the smearing method of the initial discontinuity (for the detonation to be realized we need a nonmonotonic smearing with the energy excess). Thus, we encounter a physical unremovable nonuniqueness of the Riemann problem solution. New problems with a similar nonremovable nonuniqueness have lately been discovered.

1. *The theory of elasticity with viscosity taken into account in the study of small-scale phenomena.* Nonuniqueness of self-similar problems was found in the investigation of nonlinear quasi-transverse small-amplitude waves in a weakly

anisotropic elastic media [17], which occurs in the general case of one uses only admissible discontinuities. Numerical experiments [8], [17] with viscosity taken into account showed that, depending on the details of the problem statement which are not taken into account in the simplified (hyperbolic) model, all available self-similar solutions can be realized as an asymptotics of the solution for $t \rightarrow \infty$. However, under monotonic smearing of the initial conditions the solution always follows the asymptotics of certain definite type.

2. *Nonlinear electromagnetic waves in magnetics* [10]. In this case the equations and the jump relations do not differ in the large-scale approximation from those describing quasi-transverse elastic waves. However, the structure of electromagnetic shock waves is connected with completely different mechanisms which create the dispersion of short waves. The variety of admissible discontinuities turns out for this reason completely different from the corresponding variety in the theory of elasticity. In particular, a set of separate points corresponding to discontinuities with one additional condition lie on the shock adiabatic curve. The number of these points is determined by the ratio between the dispersion and viscosity effects inside the structure and can be as large as several tens for real magnetics, thus leading to the multiple nonremovable nonuniqueness of self-similar problems.

3. As shown in [11], the problem on *the longitudinal wave propagation through the rod* has properties similar to those described in the previous example if the derivative of the rod tension with respect to its stress is nonmonotone and has at least two extremums (a minimum and a maximum). Earlier in [15] a mathematical example with exactly the same jump behavior was considered. It was based on the first order equation taking into account the dispersion and dissipation under wave propagation. The shock behavior and connected with this multiple nonuniqueness of the solutions of the self-similar problems was shown in [11] to be quite usual if the shock structure is determined by the equations with dispersion which causes the oscillation of the parameters of the medium inside the discontinuity structure.

4. Self-similar solutions can frequently be nonunique or even nonexistent if *one of the discontinuities is represented by a phase transition front*. We shall mention here only the problems dealing with ionization and recombination fronts. A systematic survey of this subject can be found in [3]. We briefly discuss only solutions describing the motion of a gas in a magnetic field where the fronts exist of the gas electrical conductivity switch-on [4]. The investigation showed that the self-similar problem can have one self-similar solution, or several ones, or none of them, depending on the choice of parameters. Numerical experiments [5] undertaken in the assumption that the only dissipative mechanism is represented by the finite gas conductivity showed that if self-similar solutions exist then the numerical solution asymptotically approaches one of them as t increases. If a self-similar solution does not exist, alternating layers of conductive and nonconductive gas originate in calculations and the number of these layers increases in time.

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